The braided monoidal structure of tube algebra representations

based on arxiv: 2511.08482

(joint w/ Makoto Yamashita)

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University of Oslo

Quantum groups Seminar (QGS)

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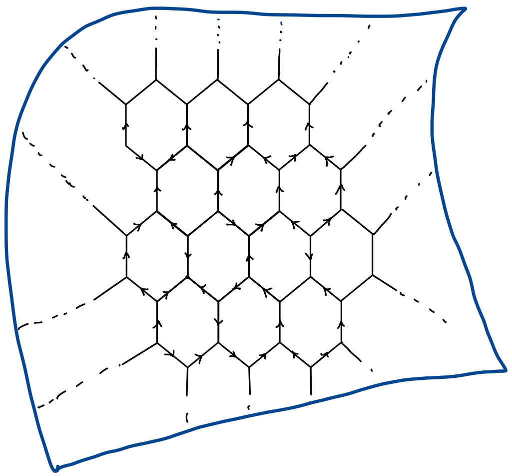
Motivation

Levin-Wen lattice models

oriented surface Σ

(storage device)

unitary fusion category $\mathcal{X}$



$\Longrightarrow$

protected space

$LW_{\mathcal{X}}(\Sigma)$

Excited states

$=$

anyons on the lattice

(given by simples in $\mathcal{Z}(\mathcal{X})$)

$\mathcal{Z}(\mathcal{X}) \xrightarrow{\simeq} \operatorname{Rep} \operatorname{Tube} \mathcal{X}$

concrete linear equivalence

Preliminaries

Definition

(i) Multitensor category: $\mathcal{X}$ locally finite $\mathbb{K}$-linear cat. + rigid monoidal str.

$$\otimes : \mathcal{X} \times \mathcal{X} \longrightarrow \mathcal{X} \;+\; \mathbb{1} \in \mathcal{X} \;+\; a \otimes \mathbb{1} \cong \mathbb{1} \otimes a \cong a \;+\; a \otimes (b \otimes c) \cong (a \otimes b) \otimes c$$

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simples

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Notation: $a \otimes b \equiv ab$

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(iii) p is spherical iff $tr^l(f) = tr^r(f) \equiv tr_a(f)$ for every $f \in End_{\mathcal{X}}(a)$

Dimensions: $\dim(a) := tr_a(id_a)$ $\qquad \dim \mathcal{A} := \sum_{a \in Irr\mathcal{A}} \dim(a)^2$

Cyclically invariant spaces of morphisms

Let $\mathcal{X}$ be a spherical multitensor category

cyclically tensorable objects $x_1, x_2, \ldots, x_n$ where $x_k \in \mathcal{X}_{i_k i_{k+1}}$ and $i_{n+1} = i_1$

$$\mathrm{Hom}_{\mathcal{X}}(1, x_1 x_2 \cdots x_n) \cong \mathrm{Hom}_{\mathcal{X}}(\overline{x_1}, x_2 \cdots x_n) \cong \mathrm{Hom}_{\mathcal{X}}(1, x_2 \cdots x_n x_1)$$

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Unit morphism for $x \in \mathcal{X}$ $\quad e_x \in H[\bar{x}\,x] \cong \mathrm{Hom}_{\mathcal{X}}(1, \bar{x}\,x) \ni \mathrm{coev}_x$

$$\begin{array}{c} \mathrm{Hom}_{\mathcal{X}}(1, \bar{x}\,x) \ni \mathrm{coev}_x \\ \cong \\ \mathrm{Hom}_{\mathcal{X}}(x, x) \ni \mathrm{id}_x \end{array}$$

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Partial traces $f \in H[y\bar{x}x]$

$$\mathrm{ptr}_x(f) := f \underset{\bar{x}x}{\circ} e_x \in H[y]$$

Full traces $f \in H[\bar{x}x]$, $x \in \mathcal{X}_{ij}$

$$\mathrm{tr}_x(f) := f \underset{\bar{x}x}{\circ} e_x \in H[1_i] \cong \mathbb{K}$$

Graphical calculus

Let $\mathcal{X}$ be a spherical multitensor category

Objects

$x \in \mathcal{X}_{ij}$

Unit

Graphical calculus

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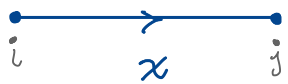
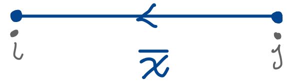
Objects 

$=$ 

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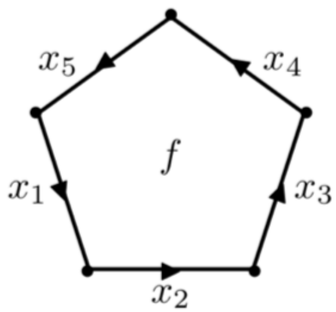
Morphisms 

$\in H[x_1 x_2 \cdots x_5]$

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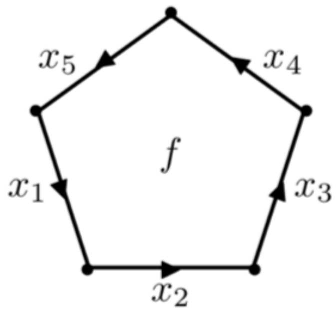

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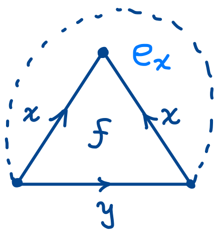
Partial traces

$ptr_x(f) =$



Full traces

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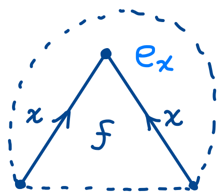
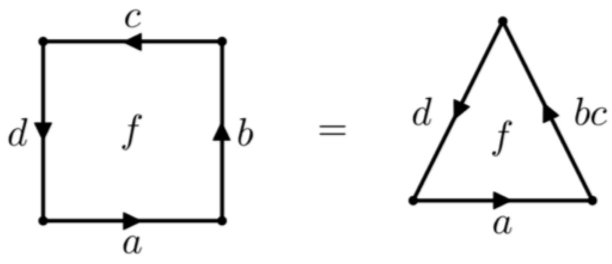


Graphical calculus

Different graphs may represent the same element in $H[x_1 \cdots x_n]$

Edge fusion:



$\in \mathrm{H}[abcd]$

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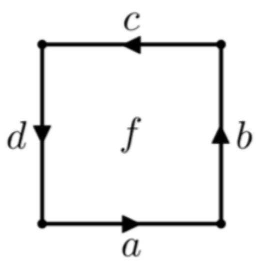

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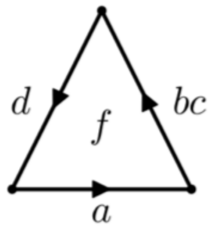

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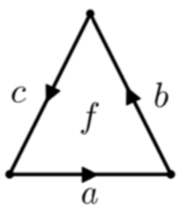

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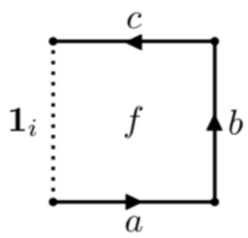

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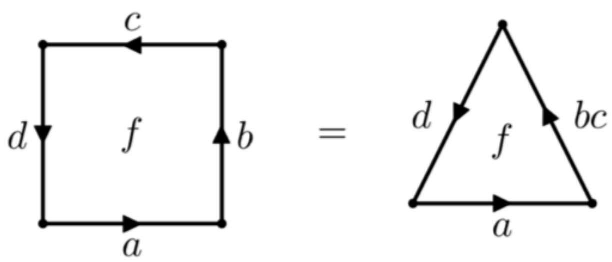

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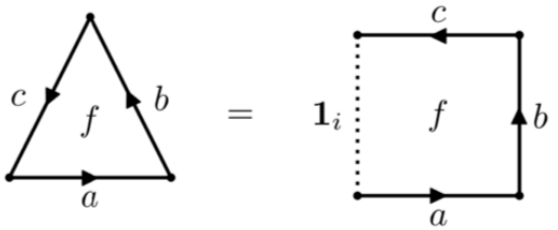

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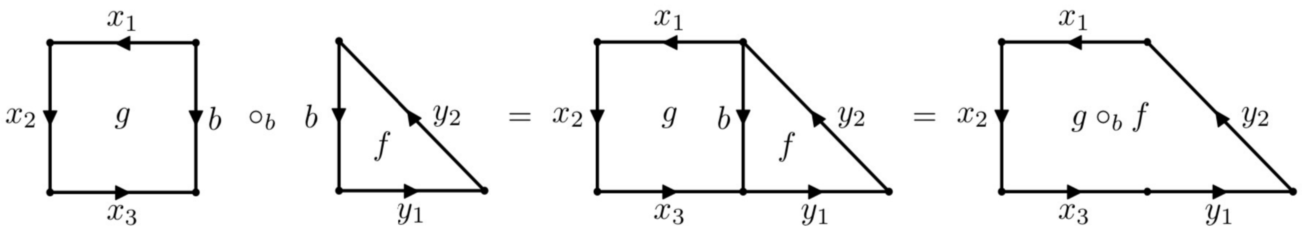
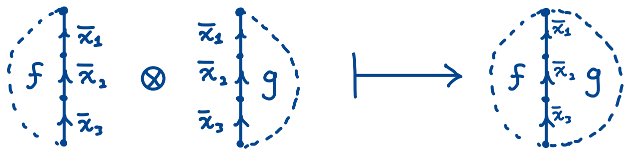
gluing:



$\in H[x_1 x_2 x_3 y_1 y_2]$

Trace Pairing

$$\langle -, - \rangle : \mathbb{H}[\overline{x}_n \cdots \overline{x}_1] \otimes \mathbb{H}[x_1 \cdots x_n] \longrightarrow \mathbb{K},$$



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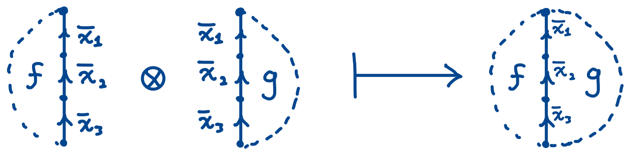
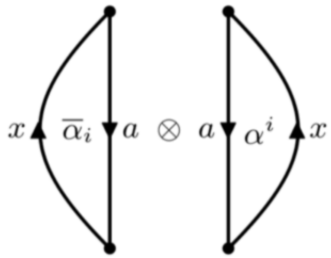


$\in \mathbb{H}[1_{in}] \cong \mathbb{K}$

<u>Definition</u>

Bases $\{\alpha^i\}$ and $\{\bar{\alpha}_j\}$ of $\mathbb{H}[xa]$ and $\mathbb{H}[\bar{a}\bar{x}]$

are paired if $\langle \alpha^i, \bar{\alpha}_j \rangle = \delta_{ij}$



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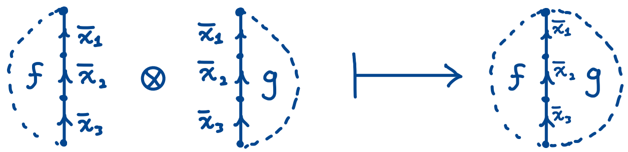

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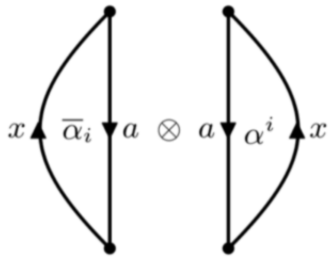
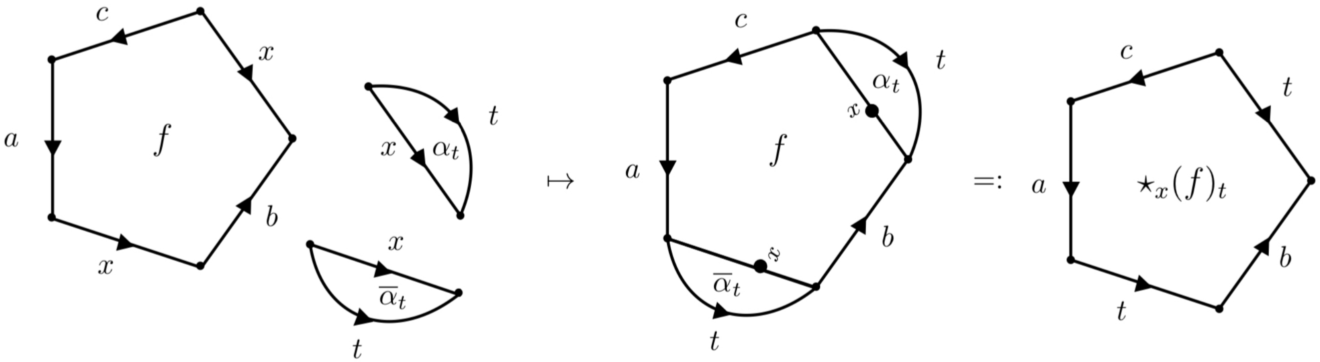


$\in H[xa] \otimes H[\bar{a}\bar{x}]$

$\Longrightarrow$ Resolution of dual edges over simple objects

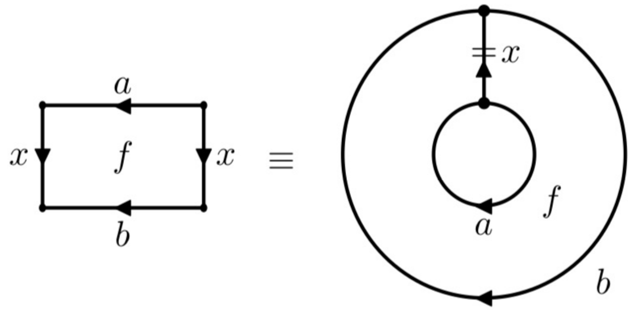
$$\star_x : H[ax b \bar{x} c] \longrightarrow \bigoplus_{t \in \mathrm{Irr}\,\mathcal{X}} H[a t b \bar{t} c]$$



Spaces of tubes

Pick $I = \coprod_{i \in \Gamma} I_i$, $\quad \operatorname{Irr} \mathcal{X}_i \subseteq I_i \subseteq \operatorname{Iso} \mathcal{X}_i$

For $a, b \in I$, $\quad T_{a;b} := \bigoplus_{x \in \operatorname{Irr} \mathcal{X}} H[\bar{b}\, \bar{x}\, a\, x]$



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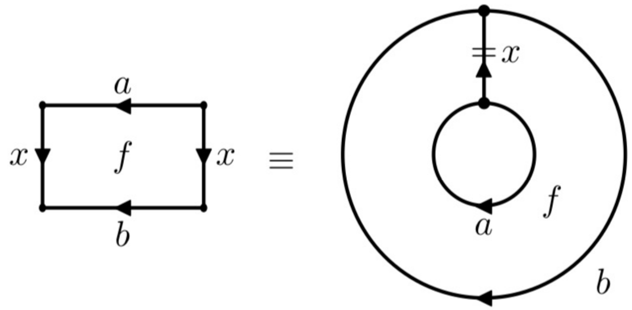
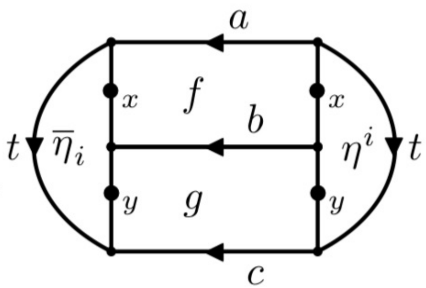


welding of tubes

for $f \in H[\bar{b}\,\bar{x}\,a\,x]$ and $g \in H[\bar{c}\,\bar{y}\,b\,y]$

$$T_{a;b} \otimes T_{b;c} \longrightarrow T_{a;c}$$

$$\star_{xy}(f \circ_b g) = \sum_{t \in \operatorname{Irr} \mathcal{X}}$$



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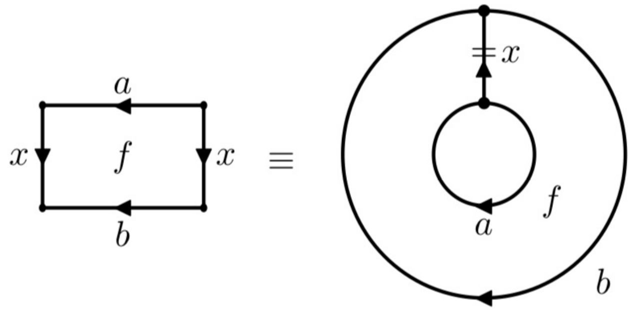
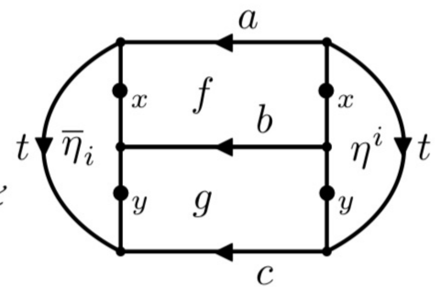


Welding of tubes $\quad$ for $f \in H[\bar{b}\,\bar{x}\,a\,x]$ and $g \in H[\bar{c}\,\bar{y}\,b\,y]$

$$T_{a;b} \otimes T_{b;c} \longrightarrow T_{a;c}$$

$$\star_{xy}(f \circ_b g) = \sum_{t \in \operatorname{Irr} \mathcal{X}}$$



Tube algebra $\quad \operatorname{Tube}(\mathcal{X}, I) := \bigoplus_{a,b \in I} T_{a;b}$

Algebras with local units and representations

Definition

(i) Algebra with local units: A $\mathbb{K}$-vector space + associative product $A \otimes A \longrightarrow A$

$\forall$ finite-dimensional $V \subset A$ $\exists$ idempotent $e \in A$ s.t. $V \subset eAe$

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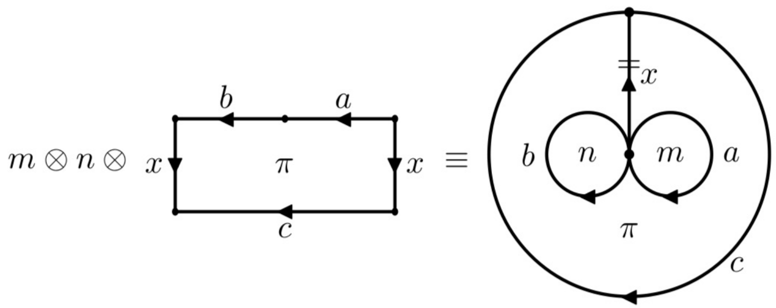
For $M \in \mathrm{Rep}\,\mathrm{Tube}(\mathcal{X}, \mathcal{I})$ $\quad M \cong \bigoplus_{a \in \mathcal{I}} M_a \quad$ with $M_a := M.e_a$

Tensor product of representations

Let $M, N \in \mathrm{Rep}\,\mathrm{Tube}(x, I)$

$$(M \tilde{\Box} N)_c := \bigoplus_{a,b \in I} M_a \otimes N_b \otimes T_{ab;c}$$



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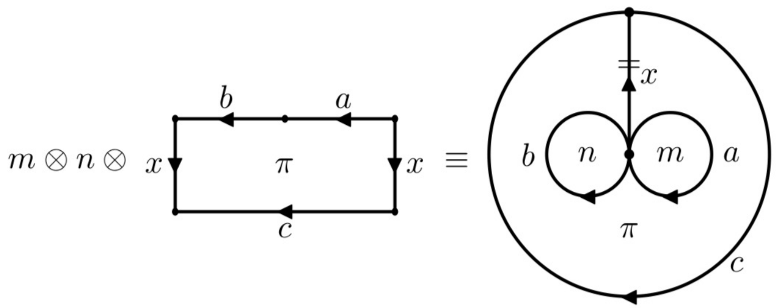
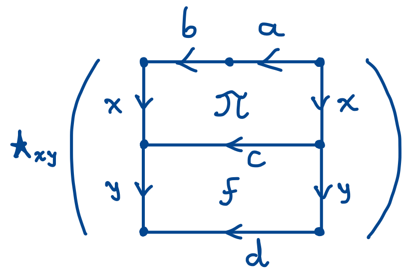


Tube (x, I) acts on $M \tilde{\Box} N$ via



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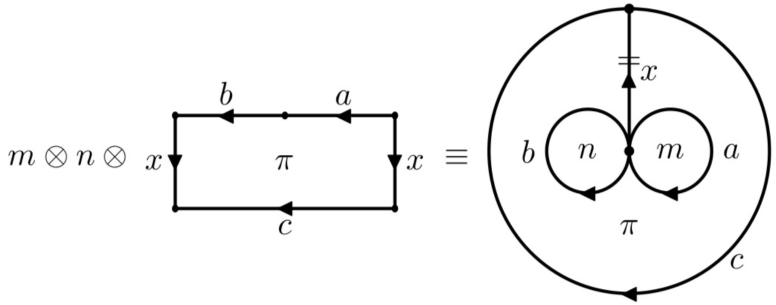

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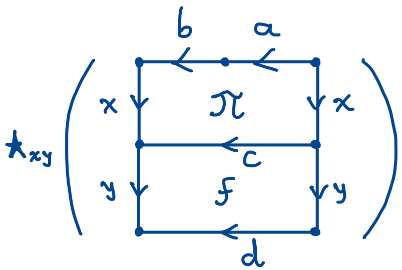

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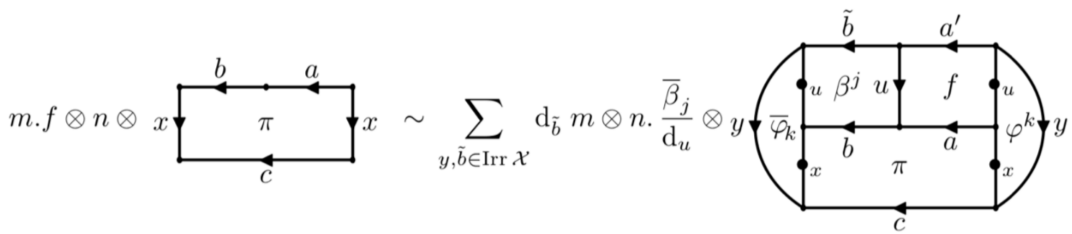
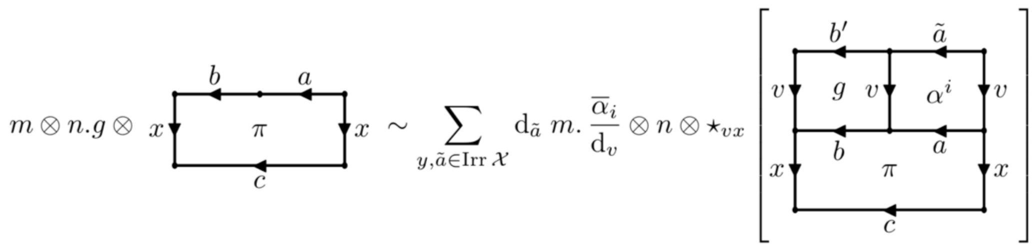


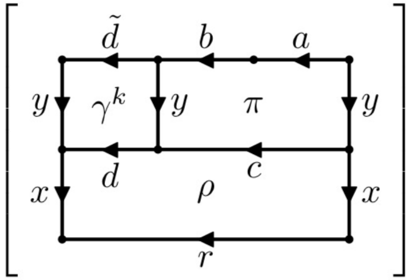


Associators

Let $M, N, L \in \mathrm{Rep}\,\mathrm{Tube}(\mathcal{X}, I)$

$$A_{M,N,L} : (M \boxtimes N) \boxtimes L \xrightarrow{\sim} M \boxtimes (N \boxtimes L)$$

$$(m \otimes n \otimes \pi) \otimes l \otimes \rho \mapsto \sum_{\tilde{d} \in \mathrm{Irr}\,\mathcal{X}} \mathrm{d}_{\tilde{d}}\, m \otimes \left(n \otimes l . \frac{\overline{\gamma}_k}{\mathrm{d}_y} \otimes \mathsf{e}_{b\tilde{d}} \right) \otimes \star_{yx}$$



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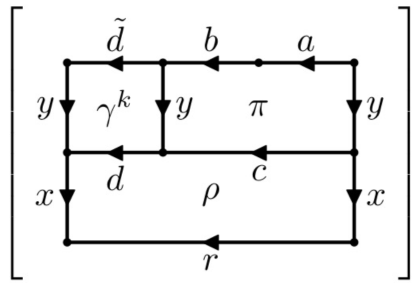

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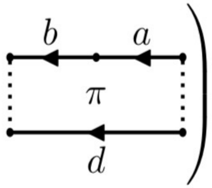
Proposition

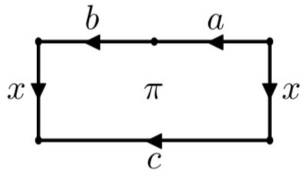
$$\mathrm{A}_{M,N,L} \left[\left(m \otimes n \otimes \right. \right.$$



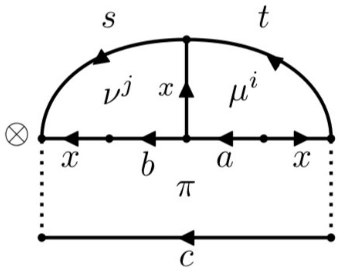
$$\left. \left. \right) \otimes l \otimes \rho \right] = m \otimes (n \otimes l \otimes \mathsf{e}_{bc}) \otimes \pi \circ_d \rho \implies \text{pentagon axiom}$$

Lemma

$$\mathrm{d}_x\, m \otimes n \otimes$$



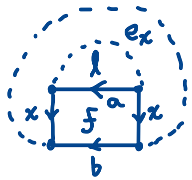
$$\sim \sum_{t,s \in \mathrm{Irr}\,\mathcal{X}} \mathrm{d}_t \mathrm{d}_s\, m.\overline{\mu}_i \otimes n.\overline{\nu}_j \otimes$$



Monoidal unit and Dualizable objects

Definition

(i) Trivial tube algebra representation

$$\mathbb{I} := \bigoplus_{a \in I} H[\bar{a}\ 1] \quad \text{with action} \quad l.f := d_x$$ 

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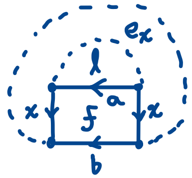

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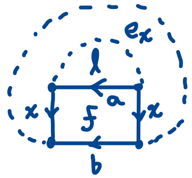

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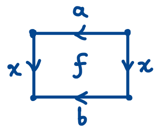
(iii) M is called locally finite if M_a finite dimensional $\forall\, a \in I$

$+ \quad \{ t \in \mathrm{Irr}\,\mathcal{X} \mid M_t \neq 0 \}$ is finite

Monoidal unit and Dualizable objects

tube involution $(-)^{\#} : T_{a;b} \longrightarrow T_{\bar{b};\bar{a}}$,



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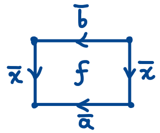

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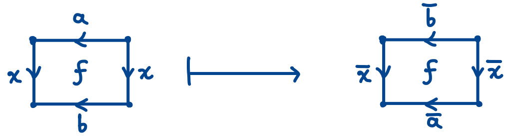
tube involution $(-)^{\#} : T_{a;b} \longrightarrow T_{\bar{b};\bar{a}}$,  $(f \cdot g)^{\#} = g^{\#} \cdot f^{\#}$

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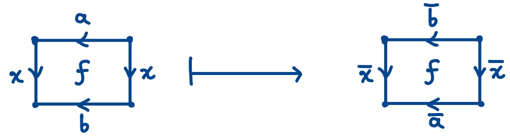

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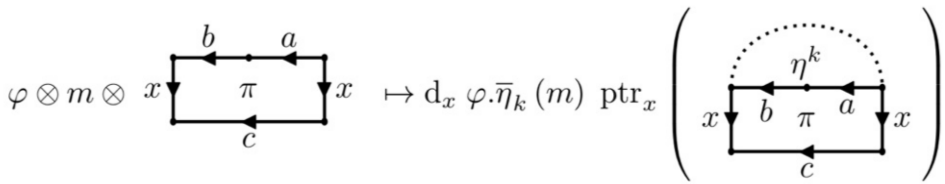

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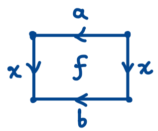
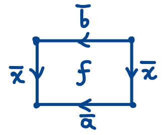
tube involution $(-)^{\#} : T_{a;b} \longrightarrow T_{\bar b;\bar a}$, 

$\longmapsto$ 

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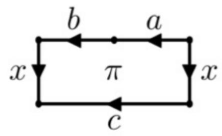
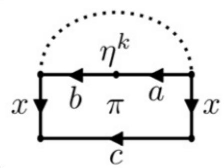
$ev_M : \overline{M} \,\Box\, M \longrightarrow \mathbb{I}$, $\quad\varphi \otimes m \otimes$ 

$\mapsto \mathrm{d}_x\ \varphi.\overline{\eta}_k\,(m)\ \mathrm{ptr}_x\Big($ 

$\Big)$

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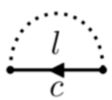
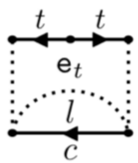
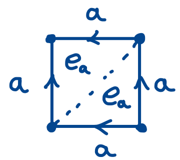
$coev_M : \mathbb{I} \longrightarrow M \,\Box\, \overline{M}$, 

$\mapsto \sum_{t\in \mathrm{Irr}\,\mathcal{X}} \mathrm{d}_t \sum_i m_t^i \otimes \overline{m}_t^i \otimes$ 

Twists and Braiding

$a \in I$, twist element $\tau_a := d_a^{-1}$ 

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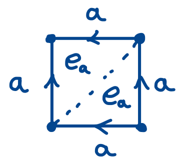
$a \in I$, twist element $\tau_a := d_a^{-1}$ 

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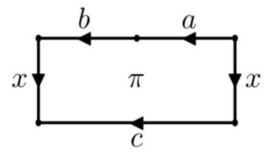
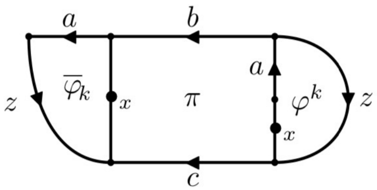
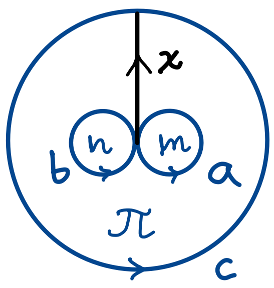
$m \otimes n \otimes$ 

$\mapsto \sum_{z \in \mathrm{Irr}\, \mathcal{X}} n \otimes m.\tau_a^{-1} \otimes z$ 



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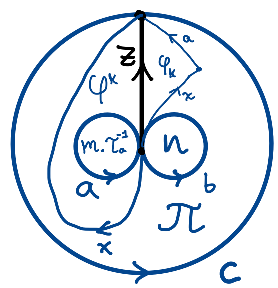

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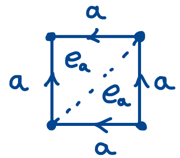
$a \in I$, twist element $\tau_a := d_a^{-1}$ 

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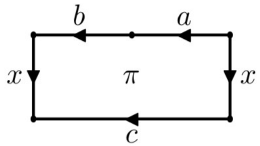
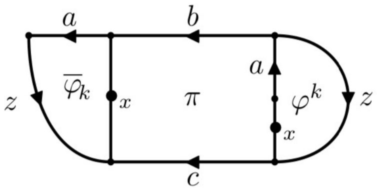
$m \otimes n \otimes x$ 

$\mapsto \sum_{z \in \mathrm{Irr}\,\mathcal{X}} n \otimes m.\tau_a^{-1} \otimes z$ 

Theorem $\mathcal{X}$ spherical semisimple multitensor category

The data $(\square, \mathbb{I}, A, B, \Theta)$ endow $\mathrm{Rep}\,\mathrm{Tube}(\mathcal{X}, I)$ with the structure of a braided monoidal category with twist.

$\mathrm{Rep}^{\text{l.f.}}\,\mathrm{Tube}(\mathcal{X}, I)$ ribbon tensor category.

Drinfeld center and Ind-completion

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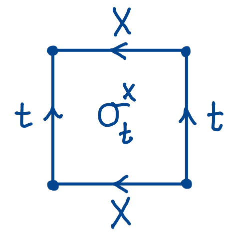
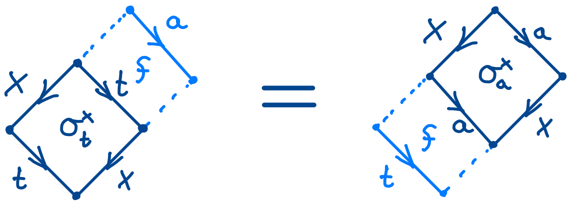
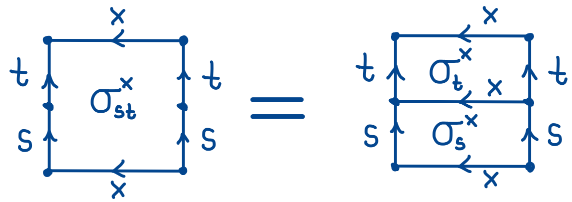
$\sigma^X_t : t\otimes X \xrightarrow{\sim} X\otimes t$ for $t\in \mathrm{Ind}\,\mathcal{X}$



Naturality:



Hexagon:



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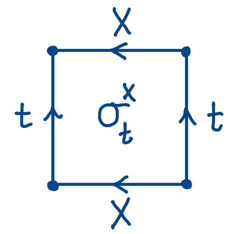
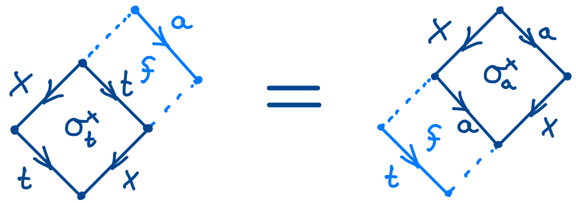
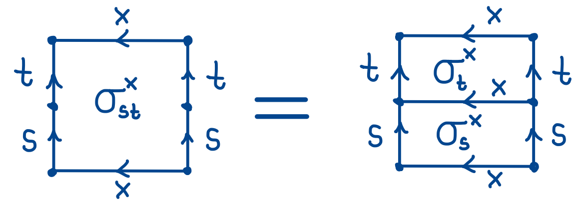
$\sigma^X_t : t\otimes X \xrightarrow{\sim} X \otimes t$ for $t \in \mathrm{Ind}\,\mathcal{X}$



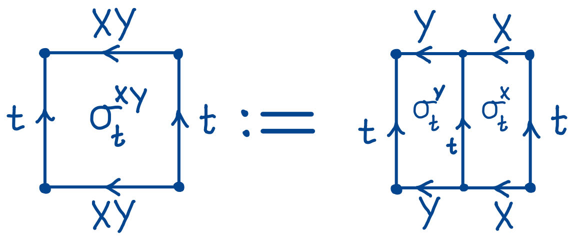
Naturality:



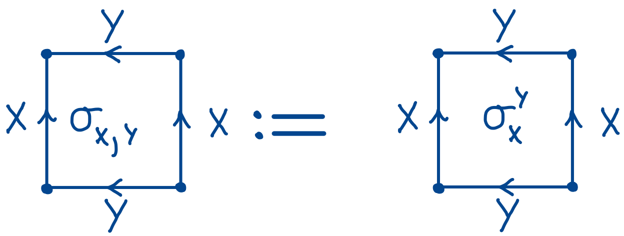
Hexagon:



Tensor product $X\otimes Y$



Braiding

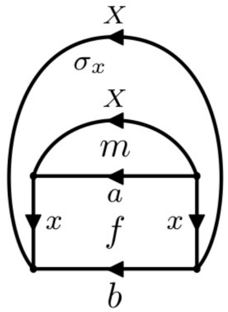


Braided monoidal Equivalence

<u>Theorem</u> [Neshveyev, Yamashita '18] There is a linear equivalence

$$E: \quad \mathcal{Z}(\operatorname{Ind}\mathcal{X}) \xrightarrow{\ \simeq\ } \operatorname{Rep}\operatorname{Tube}\mathcal{X}$$

$$(X,\sigma) \longmapsto \bigoplus_{a\in\operatorname{Irr}\mathcal{X}} H[\bar{a}X]$$

$m.f := \mathrm{d}_x$ 

Braided monoidal Equivalence

Theorem [Neshveyev, Yamashita '18] There is a linear equivalence

$$E: \quad \mathcal{Z}(\operatorname{Ind} \mathcal{X}) \xrightarrow{\simeq} \operatorname{Rep} \operatorname{Tube} \mathcal{X}$$

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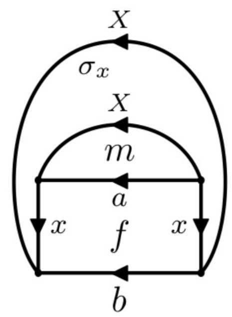
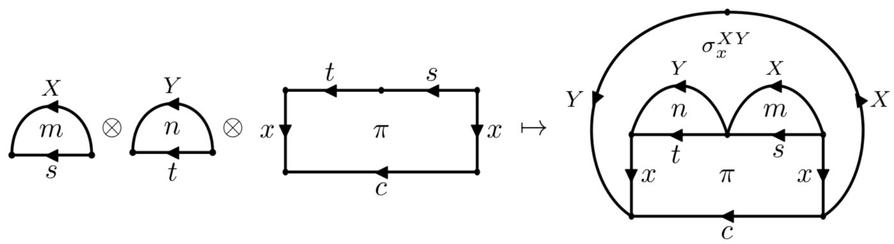
$m.f := \mathrm{d}_x$ 

Theorem [J., Yamashita '25] Given $X, Y \in \mathcal{Z}(\operatorname{Ind} \mathcal{X})$

$$\Psi_{X,Y}: E(X) \boxdot E(Y) \xrightarrow{\sim} E(X \otimes Y)$$



is a well-defined isomorphism of tube algebra representations and endows E with a braided monoidal structure.

restr. on dualizable $\quad \mathcal{Z}(\mathcal{X}) \xrightarrow{\simeq} \operatorname{Rep}^{\text{l.f.}} \operatorname{Tube} \mathcal{X}$ ribbon tensor categories

Thank

you!